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UNDERSTANDING KALMAN FILTERING AND ITS APPLICATION IN REAL TIME TRACKING SYSTEMS

J. J. Burke

Mitre Corporation

Prepared for:

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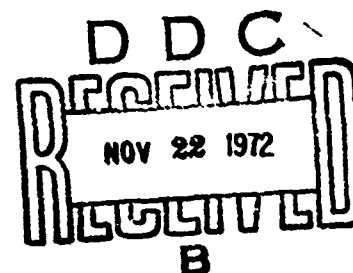ESD-TR-72-312   MTR-2417

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DEPUTY FOR SURVEILLANCE AND CONTROL SYSTEMS
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

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FOREWORD

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REVIEW AND APPROVAL

Publication of this technical report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

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13. ABSTRACT

The theory and applications of Kalman filtering have now been under development for over a decade. Kalman filtering is currently being used operationally in numerous applications such as satellite tracking systems, aircraft navigational systems and aircraft tracking systems. Because Kalman filtering is a statistical concept that is typically described in terms of state-space notation, project managers and engineers must frequently accept an abstruse set of equations while feeling insufficiently qualified to appreciate their significance or applicability. In spite of the obscure terminology and associated matrix notation that typically accompanies any discussion of Kalman filtering, the salient features of the theory are relatively straightforward and can be presented in a manner that appeals to intuition. This paper represents such an attempt with emphasis on applications to real time tracking systems.

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SECTION I

INTRODUCTION

PURPOSE

The theory and applications of Kalman filtering have now been under development for over a decade. Kalman filtering is currently being used operationally in numerous applications such as satellite tracking systems, aircraft navigational systems, and aircraft tracking systems.

Many individuals are aware of Kalman filtering and are interested in learning the salient features of the theory. In response, numerous textbooks and articles have been written on the subject. However, few if any of these treatises have been completely successful in enunciating a simple explanation of Kalman filtering that appeals to the reader's intuition. With slight exception, the discussions become immersed in awesome notation and terminology thereby dampening the reader's enthusiasm. Consequently, the system manager or project engineer must frequently accept an abstruse set of equations while feeling insufficiently qualified to appreciate their significance or applicability.

The salient features of Kalman filtering are relatively straight-forward. In spite of the obscure terminology and associated matrix notation that typically accompanies any discussion of Kalman filtering, the subject can be presented in a manner that appeals to intuition. This paper represents such an attempt with emphasis on applications to real time tracking systems.

SCOPE

Appendix A presents a derivation of the one dimensional Kalman filter equations. This derivation comprises the majority of the elements of the complete Kalman filter except that it is one dimensional and consequently does not require matrix manipulation, and there are no dynamics associated with the problem. Section II of this report discusses the more significant aspects of the Appendix A material.

Appendix B presents a general form of the Kalman multi-dimensional dynamic system equations. Each of the equations is discussed with the terminology of state-space theory being introduced to familiarize the reader with the verbiage and notation associated with the multi-dimensional, dynamic system equations. Then, having hopefully

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lessened the inscrutibility of the general Kalman filter equations, they are used to formally derive an algorithm for the real time tracking of aircraft. This algorithm is similar to that which is currently used for tracking in the 407L system and which will be used in the AWACS system. Furthermore it is anticipated that Kalman filter tracker designs similar to the derived algorithm will be proposed for the COMBAT GRANDE System. Section III of this report discusses the more significant aspects of the material that is presented in Appendix B.

Section IV concludes this report with some qualitative considerations regarding the Kalman technique.

SECTION II

ONE DIMENSIONAL KALMAN FILTER

INTRODUCTION

A one dimensional Kalman filter is derived in Appendix A. This derivation comprises the majority of the elements of the complete Kalman filter except that it is one dimensional and consequently does not require matrix manipulation, and there is no motion associated with the problem. Salient aspects of the Appendix A material are discussed in this section of the report.

COMBINATION OF ESTIMATES

The Kalman filtering process consists of combining two estimates of a variable in order to obtain an improved estimate. One of these estimates is the initial or the current value of the variable and the other is a measurement datum. The improved estimate is obtained by updating the initial or current estimate with an appropriate weighting coefficient and the measurement datum. The weighting coefficient is chosen so that the improved estimate has a minimum variance; equivalently, the improved estimate has the maximum probability of being the true value of the parameter being estimated.

For example, suppose we wish to estimate the range of a stationary or point target from a radar site. Let us assume that our initial prediction or estimate of the range of the point target is R^P. The variance associated with the range estimate R^P is Var R^P. Let us further assume that the variance associated with each range measurement R^M of the point target is Var R^M. The Kalman filtering process weighs the variance of the initial or current estimate of range with the precision or variance associated with the range measurement datum and optimally combines the two quantities, thereby obtaining an improved estimate of the range of the point target.

As derived in Appendix A, the weighting coefficient α that is used to optimally combine the initial or current estimate of range and the measurement range datum R^M to obtain an improved minimum variance range estimate R^S is defined by

$$\alpha = \frac{\text{Var } R^P}{\text{Var } R^P + \text{Var } R^M} \tag{1}$$

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where

Var R^P is the variance associated with the initial or current range estimate, and

Var R^M is the variance associated with a measured range datum.

From Appendix A, an improved range estimate R^S is obtained from the initial or current range estimate R^P, the measurement datum R^M and the weighting coefficient α in accordance with

$$R^S = R^P + \alpha (R^M - R^P) = (1-\alpha)R^P + \alpha R^M \qquad (2)$$

Examination of equation (1) reveals that it possesses attractive attributes. If the variance of the current or predicted range R^P is much larger than the variance of the measured datum R^M, the weighting coefficient α approaches unity. From equation (2) the improved range estimate R^S becomes approximately equivalent to the measured datum R^M as α approaches unity. This is of course desirable if one has considerably less confidence in the estimate R^P than in the measurement datum. On the other hand, if the variance of the current or predicted range R^P is much smaller than the variance of the measured datum R^M, the weighting coefficient α approaches zero. From equation (2), the improved range estimate R^S becomes approximately equivalent to the current or predicted range R^P. In this situation, the measured datum is essentially ignored and the new range estimate is not appreciably improved.

In addition to optimally combining two independent estimates of a parameter in order to obtain an improved range estimate, the Kalman filtering technique provides the user with a real time measure of confidence in the current estimate. This measure may be used to determine whether the current estimate is sufficiently accurate or whether additional measurements should be obtained to improve the estimate. From Appendix A, the variance of the improved estimate R^S is defined by

$$\text{Var } R^S = (1-\alpha) \text{ Var } R^P \qquad (3)$$

Replacing α in equation (3) with its definition from equation (1), this expression may be rewritten as

$$\text{Var } R^S = \frac{\text{Var } R^M \text{ Var } R^P}{\text{Var } R^P + \text{Var } R^M} = \alpha \text{ Var } R^M \qquad (4)$$

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Analysis of equations (3) and (4) reveals that they too possess attractive characteristics. If the variance of the current or predicted range R^P is much larger than the variance of the measured datum R^M, we have seen that the weighting coefficient α approaches unity and that the improved range estimate R^S becomes approximately equivalent to the measurement datum. From equation (4) we see that the corresponding variance of the improved range estimate becomes approximately equivalent to the variance of the measurement datum in this situation. On the other hand, if the variance of the current or predicted range R^P is much smaller than the variance of the measurement datum, we have seen that α approaches zero, the measurement datum is essentially ignored, and the new range estimate R^S is not appreciably improved. From equation (3) we see that the variance of the new estimate approaches Var R^P, the variance of the previous estimate for this situation. In other words, if the measurement data is imprecise relative to the predicted range, neither the new range estimate nor the variance of the new range estimate are much improved by the measurement.

Table I shows values for the weighting coefficient, the improved range estimate and the variance of the improved range estimate for the above stated conditions.

Table I

Initial Improved Estimate and Its Variance

Condition	Weighting Coefficient	Improved Estimate	Variance of Improved Estimate
Var R^P >> Var R^M	$\alpha \simeq 1$	$R^S \simeq R^M$	Var $R^S \simeq$ Var R^M
Var R^P << Var R^M	$\alpha \simeq 0$	$R^S \simeq R^P$	Var $R^S \simeq$ Var R^P

RECURSIVE FEATURE OF THE KALMAN FILTER

When another range measurement is obtained, the recursive feature of the Kalman filter permits the initial smoothed estimate R^S to be further improved based upon the measurement datum R^M and a different weighting coefficient. Like the previous smoothed estimate, the improved smoothed estimate will have minimum variance and hence maximum probability of representing the true value of the parameter being estimated. The recursive feature of the Kalman filter will be shown via an example.

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Let us assume that the initial predicted range of a stationary target is 100.0 NM. The variance associated with the predicted range is assumed to be 1.0 NM. The variance associated with each range measurement R^M is assumed to be 0.04 NM. For this example, the measurement datum is precise relative to the predicted range; consequently, the weighting coefficient α will approach unity and the initial smoothed estimate of the true range will become approximately equivalent to the measured range.

From equation (1) and using the subscript 1 to indicate that this is the initial iteration, the weighting coefficient α_1 is

$$\alpha_1 = \frac{\text{Var } R_1^P}{\text{Var } R_1^P + \text{Var } R_1^M} = \frac{1.0}{1.0 + .04} = 0.961 \tag{5}$$

Assuming the initial range measurement is three standard deviations away from the predicted range of the stationary target, R^M is 103 NM. The standard deviation of a variable is defined as the square root of the variance of the same variable. Then from equation (2), the smoothed estimate of range R^S is

$$R_1^S = R_1^P + \alpha_1 (R_1^M - R_1^P) = 100 + .961(103-100) = 102.88 \tag{6}$$

From equation (4), the variance of the smoothed estimate R^S, which indicates a confidence value associated with the current smoothed range estimate of 102.88 NM, is

$$\text{Var } R_1^S = \alpha_1 \text{ Var } R_1^M = 0.961(.04) = .038 \tag{7}$$

As expected, since the measurement datum is precise relative to the predicted range, the variance associated with the smoothed estimate is reduced to a value slightly below that of the measurement datum after one measurement.

As a result of the first iteration, we have calculated α_1 and in turn obtained an improved estimate of range namely R_1^S and an associated confidence factor in R_1^S namely Var R_1^S. When the next measurement is obtained, an analogous situation to the previous iteration exists except that we have a better estimate of predicted range namely R_1^S and a better estimate of the variance of the predicted range namely Var R_1^S. Therefore, the new weighting coefficient α_2 becomes

$$\alpha_2 = \frac{\text{Var } R_1^S}{\text{Var } R_1^S + \text{Var } R_2^M} = \frac{.038}{.038 + .04} = 0.487 \tag{8}$$

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Similarly, making use of $R_1{}^S$ in place of $R_1{}^P$ in equation (6), the next smoothed estimate of range $R_2{}^S$ is

$$R_2{}^S = R_1{}^S + \alpha_2 (R_2{}^M - R_1{}^S) \tag{9}$$

Finally, making the same substitutions in equation (7), the variance of $R_2{}^S$ is

$$\text{Var } R_2{}^S = \alpha_2 \text{ Var } R_2{}^M = .487(.04) = .019 \tag{10}$$

Comparing equations (5) through (7) with equations (8) through (10), it is apparent that a recursive procedure is developing. Each time another range measurement is available, a new and smaller weighting coefficient α is calculated. It is applied in conjunction with the most recent estimate and the measurement datum to obtain a better estimate of the true value of the parameter being measured. Thereafter a confidence factor in the most recent estimate is calculated. This confidence is expressed in terms of the variance of the estimate and may be made as small as desired by processing additional measurement data. Precise measurement data cause the improved estimate to rapidly converge to the true value of the parameter being estimated; stated alternatively, precise measurement data result in a small variance of the estimate which implies that considerable confidence may be associated with the improved estimate.

The variance of the range estimate that is calculated in any iteration of the recursive formulas must be stored to compute the weighting coefficient for the following iteration. Similarly, the most recently calculated smoothed estimate of range must be stored to compute the following range estimate. Beyond storing these two parameters plus the variance associated with each range measurement, no past data are required to be saved.

The general recursive formulas for obtaining successive values of α, R^S and $\text{Var } R^S$ that may be obtained from examination of equations (5) through (10) are

(a) Calculate the weighting coefficient

$$\alpha_N = \frac{\text{Var } R_{N-1}{}^S}{\text{Var } R_{N-1}{}^S + \text{Var } R_N{}^M}$$

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(b) Use the weighting coefficient to obtain an improved range estimate

$$R_N{}^S = R_{N-1}{}^S + \alpha_N (R_N{}^M - R_{N-1}{}^S)$$

(c) Calculate the variance of the improved range estimate

$$Var\ R_N{}^S = \alpha_N\ Var\ R_N{}^M$$

Table II shows values for the weighting coefficient α and for the variance of successive range estimates based upon four iterations of the general recursive equations, the first two iterations of which were completed in the example. It will be noted from an examination of the recursive formulas that values for α and Var R^S may be pre-calculated for any iteration before obtaining any measurements given only the variance of both the initial predicted range and the measurement data. It is not necessary to use the general recursive formula (b) in the process.

Table II

Values for α and Var R^S

Iteration	α	Var R^S
1	.961	.038
2	.487	.019
3	.322	.013
4	.245	.010

INITIALIZATION

In the example, a priori numerical values were assigned to three parameters to initialize the Kalman filtering procedure. These parameters were the predicted value of range or $R^P = 100$, the variance of the predicted range or Var $R^P = 1.0$ and the variance of successive range measurements or Var $R^M = 0.04$ NM. Thereafter the calculation

of each new estimate of the true value of range was identical to the previous iteration with a new weighting coefficient being calculated that considered the new measurement datum and all previous estimates of range.

In typical applications, a priori knowledge of the range of a stationary target is inferior to that which is derived from a single range measurement datum. Consequently it would be desirable to use the initial range measurement datum as the predicted range and to employ successive measurement data to obtain improved estimates of range. Using this technique, the variance of the predicted range would be equivalent to the variance of the initial measurement datum. For this situation, the one dimensional Kalman recursive formulas may be initialized by setting

$$\alpha_1 = 1.0$$

$$R_1^S = R_1^M$$

$$\text{Var } R_1^S = \text{Var } R_1^M$$

As the variance associated with each range measurement is assumed to be constant, successive weighting coefficients and the variances of successive range estimates may be precalculated from the general recursive formulas before obtaining any measurements as follows

$$\alpha_2 = \frac{\text{Var } R_1^S}{\text{Var } R_1^S + \text{Var } R_2^M} = \frac{\text{Var } R_1^M}{2\text{Var } R_1^M} = \frac{1}{2}$$

$$\text{Var } R_2^S = \alpha_2 \text{ Var } R_2^M = 1/2 \text{ Var } R_1^M$$

$$\alpha_3 = \frac{\text{Var } R_2^S}{\text{Var } R_2^S + \text{Var } R_3^M} = \frac{1/2 \text{ Var } R_1^M}{3/2 \text{ Var } R_1^M} = \frac{1}{3}$$

$$\text{Var } R_3^S = \alpha_3 \text{ Var } R_3^M = 1/3 \text{ Var } R_1^M$$

The sequence of weighting factors and the variances of successive range estimates that will result from using the initial range measurement datum as the predicted range R^p and from employing successive measurement data to obtain improved estimates of range are shown in Table III.

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Table III

Weighting Coefficients and Variances of Range Estimates

Iteration	α	Var R^S
1	1.0	Var R_1^M
2	1/2	Var $R_1^M/2$
3	1/3	Var $R_1^M/3$
4	1/4	Var $R_1^M/4$
.	.	.
.	.	.
.	.	.
N	1/N	Var R_1^M/N

Table III shows that the estimated value of range R^S becomes more and more reliable as successive measurement data are obtained and processed. Any desired degree of reliability may be attained by processing additional measurement data. This is of course a well known result from elementary statistics.

CORRELATION

In estimating the target's range, the general recursive formulas will employ measurement data that should not be associated with the target unless a maximum deviation between the predicted and the measured range is imposed. In the example where a priori knowledge was available, the predicted range of the target was assumed to be 100 miles, the standard deviation of the predicted range was assumed to be 1.0 NM and the standard deviation of each measurement datum was assumed to be 0.2 NM. Specifying that the standard deviation of the measurement data is 0.2 NM indicates that measurement data associated with the target will generally be within three standard deviations or 0.6 NM of the true range of the target. Likewise, specification that the standard deviation of the prediction error is 1.0 NM indicates that true target range is within approximately three standard deviations or 3 NM of the predicted range.

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Making use of these parameters and recalling that the variance of a variable is the square of the standard deviation of the variable, an optimum choice of the maximum deviation between a measurement datum and the predicted range may be constructed. Since the variance of the difference between the predicted and the measured range is the sum of the variances of the predicted and measured ranges, an optimum filter will correlate an initial measurement datum that satisfies

$$|R^M - R^P| \leq 3(\text{Var } R^M + \text{Var } R^P)^{\frac{1}{2}} = 3(1.04)^{\frac{1}{2}} \tag{11}$$

Successive measurement data must satisfy the following inequality in order to be processed

$$|R_N^M - R_N^S| \leq 3(\text{Var } R_N^M + \text{Var } R_N^S)^{\frac{1}{2}} \tag{12}$$

As Var R_N^S decreases with successive iterations as shown in Tables II and III, the filter employs decreasing correlation bounds as a more and more reliable estimate of the target's range is obtained.

SECTION III

MULTI-DIMENSIONAL KALMAN FILTER

INTRODUCTION

A general form of Kalman's multi-dimensional, dynamic system equations is presented in Appendix B. The concepts of state variable or vector, transition matrix, measurement matrix, covariance matrix and other notation of state-space theory are introduced to familiarize the reader with the terminology associated with the multi-dimensional, dynamic system equations. Analogies between the general Kalman equations and the one dimensional non-dynamic filter are presented.

This section of the report presents the salient aspects of the material that is presented in Appendix B. Analogies between the multi-dimensional tracking filter and the one dimensional filter are indicated.

SYSTEM MODELING

Kalman filtering theory as applied to dynamic systems requires the definition of a mathematical model describing the physical phenomena associated with the estimation problem. Procedures used in this process are rather subjective in nature; therefore, some general considerations and guidelines relative to defining a model that is applicable to real time Kalman tracking systems will be given. The tracking systems to be considered use sensors which provide measurements of range and azimuth. The aircraft to be tracked are assumed to be capable of maneuvering.

The mathematical model describing the physical process associated with the estimation problem should express the state variables at one time as a function of the state variables at a previous time. State variables may be defined as the minimum set of variables that provides full knowledge of the system's behavior.

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A model describing the position, speed and acceleration state variables for an aircraft may be expressed as

$$R^P = R^S + \dot{R}^S \Delta + \ddot{R}^S \Delta^2/2$$

$$\dot{R}^P = \dot{R}^S + \ddot{R}^S \Delta$$

$$\ddot{R}^P = \ddot{R}^S \qquad\qquad\qquad (13)$$

$$\psi^P = \psi^S + \dot{\psi}^S \Delta + \ddot{\psi}^S \Delta^2/2$$

$$\dot{\psi}^P = \dot{\psi}^S + \ddot{\psi}^S \Delta$$

$$\ddot{\psi}^P = \ddot{\psi}^S$$

where the superscript P denotes the predicted state variables, where the superscript S denotes the smoothed state variables from the previous iteration and where Δ is the time since the last smoothing.

Since computer storage and particularly processing requirements increase alarmingly with the number of state variables, designers are generally motivated to seek simplifications to the modeling equations that will result in minimal degradation in the estimation of the major state variables. Consequently, Kalman aircraft tracking filters usually employ a constant speed, straight line model of aircraft motion. Such a model results in a filter that is incapable of tracking maneuvering aircraft; therefore the filter, as derived from Kalman's equations with a constant velocity straight line model of aircraft motion, is augmented by a maneuver response logic. The maneuver response logic will be discussed in a subsequent section of this report.

The coordinate system that is used to define the model of aircraft motion is another factor to be considered in applying Kalman filter theory. Selection of an inappropriate coordinate system typically results in a complex algorithm being derived from the Kalman equations which requires inordinate storage and processing requirements. Furthermore and as noted in Appendix B, the use of a non-judicious coordinate system may be incompatible with the hypothesized model of motion.

For these reasons the model of aircraft motion that is generally used in deriving a Kalman tracking algorithm assumes that the aircraft is not accelerating, employs a rectilinear coordinate system and may be expressed by

$$X^P = X^S + \dot{X}^S \Delta$$

$$\dot{X}^P = \dot{X}^S$$

$$Y^P = Y^S + \dot{Y}^S \Delta \tag{14}$$

$$\dot{Y}^P = \dot{Y}^S$$

Equation (14) may be written in matrix notation as

$$
\begin{bmatrix} X^P \\ \dot{X}^P \\ Y^P \\ \dot{Y}^P \end{bmatrix}
=
\begin{bmatrix} 1 & \Delta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} X^S \\ \dot{X}^S \\ Y^S \\ \dot{Y}^S \end{bmatrix} \tag{14a}
$$

Equation (14a) is called the predicted state variable equation with the dynamics of the system being modeled by the 4 x 4 matrix which is called the transition matrix.

In addition to decreasing the number of state variables and employing a suitable coordinate system, another technique is commonly employed to further reduce the processing and storage requirements for the derived algorithm. This technique partitions the state variables into several groups; relative to equations (14) or (14a), the state variables X and $\dot{X}$ are treated independently of those for Y and $\dot{Y}$. Then, an X component Kalman filter tracking algorithm is derived with the use of the X component model of motion as is shown in Appendix B. An independent but analogous Y component algorithm is similarly derived. This technique results in reducing the 4 x 4 matrix representation of the hypothesized model of aircraft motion to the following 2 x 2 formulation.

$$
\begin{bmatrix} X^P \\ \dot{X}^P \end{bmatrix}
=
\begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix}
\begin{bmatrix} X^S \\ \dot{X}^S \end{bmatrix} \tag{15}
$$

This legerdemain results in an identical matrix equation for the
Y state variables. Uncoupling X and Y in formulating the predicted
state variable equation yields independent X and Y tracking algorithms
that must be solved each iteration. However, the processing and
storage requirements for the two algorithms are less than those which
would result from coupling the X and Y state variables because the
cross correlations between groups such as XY, $\ddot{X}\ddot{Y}$ and $X\dot{Y}$ are ignored.
Another useful aspect of uncoupling the X and Y state variables is
that it will only be necessary to discuss the tracking algorithm in
terms of the X state variables. Replacing X by Y yields the analogous
Y component discussion.

KALMAN TRACKING FILTER

Kalman filtering as applied to aircraft tracking consists of com-
bining two estimates of a track's parameters in order to obtain an
improved estimate. One of these estimates is the predicted track
parameters and the other is a measurement datum.

In the one dimensional filter example that was presented in
Section I, three quantities were required to initialize the recursive
process. They were the expected or predicted value of range prior to
any measurements, the variance of the predicted range estimate and
the variance of the measurement datum. Similarly, in the multi-
dimensional tracking filter, analogous quantities are required for
initialization.

The quantities that are analogous to the expected value of range
are the two components of the state vector, namely X^S and $\dot{X}^S$. Cor-
responding to the variance of the predicted range are the elements of
the covariance matrix namely Var X^S, Cov $X\dot{X}^S$ and Var $\dot{X}^S$. These ele-
ments of the covariance matrix are defined in Appendix B. Finally,
analogous to the variance of the one dimensional range measurement
datum are the cartesian coordinate variances derived from the measure-
ment datum (R, ψ). The Kalman tracking algorithm weighs the variance
of the initial ˙ current estimate of the track's parameters with the
degree of prec: ion or variance of the measurement datum and optimally
combines these two quantities to obtain an improved estimate of the
track's parameters.

Let us assume that the coordinates of an aircraft derived from two
scans of a radar are used to initiate a track. The initial operator entry
of the aircraft's position at time t_0 is X_0^m, Y_0^m and the following opera-
tor entry at time t_1 is X_1^m, Y_1^m where Δ is the time between the two

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operator entries namely $t_1 - t_0$. The initial smoothed state variables X^S and $\dot{X}^S$, which correspond to the initial track position and speed are defined by

$$X^S = X_1{}^m$$
$$\dot{X}^S = (X_1{}^m - X_0{}^m)/\Delta$$

(16)

Equations (16) position the track associated with the aircraft being tracked at the coordinates of the later measurement datum and estimate the aircraft's speed from the positions of the two measurement data and the time interval between the data.

Like the one dimensional example of Appendix A, the Kalman tracking algorithm provides a real time measure of confidence in the current estimate of the aircraft's state variables. Since equations (16) calculate the initial estimates of the track's parameters from the two measurement data used in the initiation process, one's confidence in the initial estimates should be based upon the precision or variance of the same measurement data. As derived in Appendix B, the initial values of the covariance elements associated with the state variables X and $\dot{X}$ are defined by

$$\text{Var } X^S = \text{Var } X^m$$
$$\text{Var } \dot{X}^S = 2 \text{ Var } X^m/\Delta^2$$
$$\text{Cov } X\dot{X}^S = \text{Var } X^m/\Delta$$

(17)

Analysis of equations (17) reveals that they possess attractive characteristics. The first equation states that the variance of the initial estimate of the track's X position is equal to the variance of the X component of the measurement datum that is used in the initiation process. The second equation indicates that the variance of the initial estimate of the track's X component of velocity is determined from the variances of both measurement data used in the initiation process as well as the time interval between the data. If the variance of the measurement data should increase or if the time separation between the data used in the initiation process should decrease, the variance associated with the initial speed estimate will increase; i.e., the reliability of the estimate decreases. The third equation is a measure of the degree of correlation between the state variables X^S and $\dot{X}^S$. If these variables were completely independent and hence uncorrelated, Cov $X\dot{X}^S$ would equal zero. As X^S and $\dot{X}^S$ are calculated from the same measurement data they are dependent and their covariance is not zero.

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Equation (17) is a measure of confidence in the initial estimate of the track's state variables in terms of the X component variance of the measurement data that are used in the initiation process. It is therefore of interest to determine how the value of Var X^m is obtained from measurement data which provide range and azimuth information. From Appendix B, Var X^m is obtained from the relationship

$$\text{Var } X^m = \text{Var } R \text{ Sin}^2\psi + R^2 \text{ Var } \theta \text{ Cos}^2\psi \qquad (18)$$

where (R, ψ) are the coordinates of the measurement datum and where Var R and Var θ are based upon a priori knowledge of the sensor's characteristics.

For measurement data with an azimuth of $\psi = 90$ degrees, equation (18) expresses the fact that the variance of the X component of the measurement datum equals the variance of range. For measurement data with an azimuth of $\psi = 0$ degrees, equation (18) states that the X component variance of the measurement datum equals the product of range squared times the azimuth variance. These results, as well as the results for intermediate angles, are consistent with intuition.

In the one dimensional example of Section I, the smoothed variance from one iteration was used as the predicted variance for the following iteration. This technique was satisfactory since there were no dynamics associated with the problem and consequently the variance associated with a smoothed estimate was constant from one iteration to the next.

For the two dimensional tracking example, the analogous smoothed covariance matrix associated with the initiation process may not be used in the following iteration as the reliability of the positional estimate decreases as the time between iterations increases. This occurs because the error in velocity propagates into a larger positional error as the time interval between iterations increases. To compensate for the decreased reliability of the positional estimate as the time between iterations increases, the initial smoothed covariance elements must be extrapolated to the time of the following iteration. From Appendix B, the predicted covariance elements are calculated from the smoothed covariance elements associated with the initiation process and the time interval Δ between the initial and the current iteration according to the relationships

$$\text{Var } X^P = \text{Var } X^S + 2\Delta \text{ Cov } X\dot{X}^S + \Delta^2 \text{ Var } \dot{X}^S$$

$$\text{Var } \dot{X}^P = \text{Var } \dot{X}^S \qquad (19)$$

$$\text{Cov } X\dot{X}^P = \text{Cov } X\dot{X}^S + \Delta \text{ Var } \dot{X}^S$$

The superscript P indicates the predicted covariance elements whereas the superscript S denotes the smoothed covariance elements associated with the initiation process.

Correlation

In most tracking logics, radar measurement data are paired with tracks via gross distance checks followed by more precise correlation checks. The gross distance check associates all returns that lie within a given area centered at the predicted track position with that track. This check is intended to eliminate further attempts at correlation of data with tracks which represent aircraft that could not possibly have produced the data. More exacting correlation tests are then applied to the measurement data and data that pass these tests are defined as correlated data.

Non-maneuver and maneuver gates or small and large search areas are defined. Presence of a measurement datum within the non-maneuver gate centered at the predicted track position constitutes a non-maneuver correlation. Lack of a measurement datum within the non-maneuver gate and presence of a datum within the maneuver gate that is centered at or behind the predicted track position constitutes declaration of an aircraft maneuver. Non-maneuver and maneuver gates are typically circles, rectangles or annular wedges.

These concepts are applicable both to the classical aircraft tracking algorithms that are used in the SAGE and the BUIC systems and the more recent Kalman filter algorithms which are used in the 407L and the AWACS systems. The fundamental distinction between the Kalman and the classical correlation techniques is that the gate sizes are dynamically calculated and vary from one iteration to the next in a Kalman filter. These gate sizes are calculated based upon the variance of the predicted track position and the variance of the measurement datum.

Non-Maneuver Gates

Non-maneuver gates must be of sufficient size to provide a high probability of correlating measurement data with a track for non-maneuvering aircraft. Therefore with a Kalman tracker, a non-maneuver gate correlation of a measurement datum with a track occurs in the X dimension when the measurement datum falls within the track's non-maneuver correlation gates as defined by

$$|X^M - X^P| \quad < \quad K(\text{Var } X^P + \text{Var } X^M)^{\frac{1}{2}} \quad = \quad C \tag{20}$$

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X^M is the cartesian coordinate of the measurement datum and X^P is the predicted X coordinate of the track's position. The term Var X^P, which is defined by equation (19), represents the variance associated with the predicted track position. Var X^M, which is defined by equation (18), represents the variance in the X component of the measurement datum. The variance of the difference between the two terms on the left hand side of equation (20), namely the measured and the predicted track positions, is the sum of the variance of the predicted track position and that of the measured track position or Var X^P plus Var X^M. Equation (20), therefore, indicates that a measurement datum within K standard deviations of the difference between the measured and the predicted track positions will correlate with the track. K typically assumes a value of approximately 3.0.

Maneuver Gates

Maneuver gates must be of sufficient size to provide a high probability of correlating measurement data with a track when an aircraft executes a maneuver. Therefore, with a Kalman tracker, a maneuver gate correlation of a measurement datum with a track in the X dimension occurs when a non-maneuver correlation as defined by equation (20) has not occurred and when a measurement datum falls within the track's maneuver gates as defined by

$$|X^M - X^S| \quad < \quad K(C + A\ \Delta^2/2 + V\Delta) \tag{21}$$

The rationale for the maneuver gate check as defined by equation (21) is to impose an upper bound on the X component of a measurement datum that will be permitted to correlate with the track, assuming that the corresponding aircraft has linearly accelerated or executed a turn during the intervening time interval Δ. X^S is the smoothed track position from the initial or last iteration; note that X^P is not used in equation (21) as it was in equation (20) since the use of the predicted cartesian coordinate would tacitly assume straight line constant speed aircraft motion over the interval from the time of the initial or last track smoothing to the current correlation time. C is the non-maneuver gate tolerance that was used in equation (20). Δ is the time interval between the current correlation time and the time of the last track smoothing. The term $A\ \Delta^2/2$ provides for the possibility that the aircraft has linearly accelerated since the time of the last track smoothing. A typical value for A would be 13.3 NM/min^2 corresponding to a .7g linear acceleration. The term $V\Delta$ provides for the possibility that the aircraft has turned since the time of the last track smoothing. Providing for a worst case situation, it is assumed that the aircraft has turned almost instantaneously at the time of last smoothing and proceeded with a speed of V that would typically be equivalent to the current track speed plus approximately 3.0 times the current standard deviation of the track's speed.

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Track Smoothing

In the one dimensional example, a weighting coefficient α was used to optimally combine the initial estimate of range and a measurement datum. Analogous weighting coefficients are employed in the tracking algorithm. From Appendix B, the appropriate positional weighting coefficient α and speed weighting coefficient β that are used to optimally combine the initial estimate of the track's parameters and a measurement datum to obtain an improved estimate of the track's parameters are

$$\alpha = \frac{\text{Var } X^P}{\text{Var } X^P + \text{Var } X^M}$$

$$\beta = \frac{\text{Cov } X\dot{X}^P}{\text{Var } X^P + \text{Var } X^M}$$

(22)

It will be noted that the speed weighting coefficient β as defined above is in units of inverse time.

The improved smoothed track parameters designated X_2^S and $\dot{X}_2^S$ are obtained from the initial track parameters that are extrapolated to the time of the second iteration, the measurement datum from the second iteration and the weighting coefficients in accordance with the relationships

$$X_2^S = (X_1^S + \dot{X}_1^S \Delta) + \alpha(X_2^m - X_2^P) = X_2^P + \alpha(X_2^m - X_2^P)$$

$$\dot{X}_2^S = \dot{X}_1^S + \beta(X_2^m - X_2^P) = \dot{X}_2^P + \beta(X_2^m - X_2^P)$$

(23)

Examination of equations (22a) and (23a) reveals that they are identical to equations (1) and (2) of the one dimensional example. Consequently, all of the discussion relative to equations (1) and (2) is equally appropriate to equations (22a) and (23a). The implications of equations (22b) and (23b) are less obvious, however, and will be discussed via numerical examples when the recursive feature of the Kalman tracker is considered.

In addition to optimally combining two independent estimates of the track's parameters to obtain improved estimates, the Kalman tracker provides a real time measure of confidence in the current

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estimates of the track's parameters. From Appendix B, the smoothed covariance elements associated with the improved estimates of the track's parameters are defined by

$$\text{Var } X^S = \alpha \text{ Var } X^m$$

$$\text{Var } \dot{X}^S = \text{Var } \dot{X}^P - \beta \text{ Cov } X\dot{X}^P \tag{24}$$

$$\text{Cov } X\dot{X}^S = \beta \text{ Var } X^m$$

Examination of equation (24a) reveals that it is identical to equation (4) of the one dimensional example. Since equations (24b) and (24c) do not have analogous one dimensional counterparts, these equations will be discussed via numerical examples when the recursive feature of the Kalman tracker is considered.

Equations (16), (17), and (18) constitute the initialization process for the Kalman tracking algorithm. Equations (19) through (24), with the use of equation (18), constitute the X coordinate Kalman prediction, correlation, and smoothing filter. Substituting Y for X in each of these equations results in the Y component filter equations.

RECURSIVE FEATURE OF THE KALMAN TRACKER

When successive measurement data are obtained, the recursive feature of the Kalman tracker permits improved estimates of the track's parameters to be calculated based upon the measurement data and different weighting coefficients. Like the previous smoothed estimate, the improved estimate will have minimum variance and, hence, maximum probability of representing the true value of the track's parameters. The recursive feature of the Kalman tracker will be shown via several examples.

Unity Blipscan X Axis Aircraft Motion

Let us assume that an aircraft is flying along the X axis of a rectilinear coordinate system centered at the radar site at a speed of 720 knots or 2.0 NM per 10.0 seconds. Let us further assume that a track is initiated based upon two measurement data associated with the aircraft from consecutive scans of the radar which is turning at

a rate of 6 RPM, so that the time interval Δ between the two measurement data is 10.0 seconds. The initial measurement datum at time $t=0$ and the following measurement at time $t=10$ seconds are respectively

$$R_0, \psi_0 = 200.1 \text{ NM, 90 degrees or } X_0^m = 200.1 \text{ NM}$$

$$R_1, \psi_1 = 197.9 \text{ NM, 90 degrees or } X_1^m = 197.9 \text{ NM}$$

A priori knowledge of the radar's characteristics indicates that the range variance is 0.25 NM squared and the azimuth variance is 3 ACPs squared, where an ACP is one 4096th part of a circle or 0.088 degrees.

From equation (16) the initial smoothed state variables X^S and $\dot{X}^S$, which correspond to the initial track position and speed, are defined by

$$X^S = X_1^m = 197.9 \text{ NM}$$

$$\dot{X}^S = \frac{X_1^m - X_0^m}{\Delta} = \frac{197.9 - 200.1}{10} = \frac{-2.2 \text{NM}}{10 \text{ sec}} = -792 \text{ knots}$$

(25)

The initial values of the covariance elements associated with these estimates of track position and speed are determined from equations (17) and (18) and are defined by

$$\text{Var } X^S = \text{Var } X^m = \text{Var } R = .06250 \text{ since } \psi = 90 \text{ degrees}$$

$$\text{Var } \dot{X}^S = 2 \text{ Var } X^m/\Delta^2 = .00125$$

$$\text{Cov } X\dot{X}^S = \text{Var } X^m/\Delta = .00625$$

(26)

From equation (19), the predicted covariance elements are calculated from these smoothed covariance elements and the time interval-$\Delta = 10.0$ seconds from the previous to the current measurement datum according to the relationships

$$\text{Var } X^P = \text{Var } X^S + 2\Delta\text{Cov } X\dot{X}^S + \Delta^2 \text{ Var } \dot{X}^S = .31250$$

$$\text{Var } \dot{X}^P = \text{Var } \dot{X}^S = .00125$$

$$\text{Cov } X\dot{X}^P = \text{Cov } X\dot{X}^S + \Delta \text{ Var } \dot{X}^S = .01875$$

(27)

This result may be verified by substituting the values for the smoothed covariance from equation (17) into equation (19), thereby obtaining the result that

$$\text{Var } X^P = 5 \text{ Var } X^m = 5(.0625) = .31250$$

$$\text{Var } \dot{X}^P = 2 \text{ Var } X^m / \Delta^2 = .00125$$

$$\text{Cov } X\dot{X}^P = 3 \text{ Var } X^m / \Delta = .01875$$

(28)

From equation (20), a measurement datum is classified as a non-maneuver correlation if it falls within C miles of the predicted track position where C is defined by

$$C = K(\text{Var } X^P + \text{Var } X^m)^{\frac{1}{2}} = 3.0 \ (.3125 + .0625)^{\frac{1}{2}} = 1.837 \text{ NM}$$

(29)

From equation (22), the positional weighting factor α and the speed weighting factor β are defined by

$$\alpha = \frac{\text{Var } X^P}{\text{Var } X^P + \text{Var } X^m} = \frac{.3125}{.3125 + .0625} = .83$$

(30)

$$\beta = \frac{\text{Cov } X\dot{X}^P}{\text{Var } X^P + \text{Var } X^m} = \frac{.01875}{.3125 + .0625} = .05$$

Since the aircraft is proceeding at 2.0 NM/10 seconds, its true coordinates at time $t = 20$ seconds are $R_2 = 196$ NM and $\psi_2 = 90$ degrees. Assuming that the range measurement is two standard deviations or 0.5 NM away from the true aircraft location, $R_2^M = 196.5$ NM and $\psi_2 = 90$ degrees. Substituting from equation (25) into equation (23), the improved estimates of the aircraft's position and speed are

$$X_2^P = X_1^S + \dot{X}_1^S \Delta = 197.9 - (2.2/10) \ 10 = 195.7 \text{NM}$$

$$\dot{X}_2^P = \dot{X}_1^S = -2.2 \text{NM}/10 \text{ sec} = -792 \text{ knots}$$

(31)

$$X_2^S = X_2^P + \alpha(X_2^m - X_2^P) = 195.7 + .83(196.5 - 195.7) = 196.3$$

$$\dot{X}_2^S = \dot{X}_2^P + \beta(X_2^m - X_2^P) = \frac{-2.2}{10} + .05(0.8) = -2.16 \text{NM}/10 \text{ sec} = -777.6 \text{ knots}$$

From equation (24), the measures of confidence in the current estimates of the track's parameters are

$$\text{Var } X^S = \alpha \text{ Var } X^m = .83(.0625) = .05208$$

$$\text{Var } \dot{X}^S = \text{Var } \dot{X}^P - \beta \text{ Cov } X\dot{X}^P = .00125 - .05(.01875) = .000313 \qquad (32)$$

$$\text{Cov } X\dot{X}^S = \beta \text{ Var } X^m = .05(.0625) = .003125$$

Equations (25) and (26) have yielded initial estimates of the track's parameters along with their associated measures of confidence. Equations (27) and (28), which represent the start of the first complete iteration, extrapolate the measures of confidence to the time of that iteration and equation (29) employs this data along with the variance of the measurement data to calculate the X correlation distance. Equations (30) and (31) calculate the weighting coefficients and use the coefficients to obtain improved estimates of the track's parameters. Finally, equation (32) computes measures of confidence associated with the most recent estimates of the track's parameters. Each time another measurement datum is available, equations (27) through (31) are repeated.

The smoothed covariance elements from one iteration must be saved to calculate the predicted covariance elements for the following iteration. Similarly, the current smoothed track parameters must be stored to calculate the following improved estimate. Beyond storing these parameters, no past data is required to be saved.

The general recursive formulas for obtaining successive values of the predicted covariance matrix, the correlation distance, the weighting coefficients, the improved estimates of the track's parameters and their associated measures of confidence are

(a) Calculate the predicted covariance elements

$$\text{Var } X_N^P = \text{Var } X_{N-1}^S + 2\Delta \text{ Cov } X\dot{X}_{N-1}^S + \Delta^2 \text{ Var } \dot{X}_{N-1}^S$$

$$\text{Var } \dot{X}_N^P = \text{Var } \dot{X}_{N-1}^S$$

$$\text{Cov } X\dot{X}_N^P = \text{Cov } X\dot{X}_{N-1}^S + \Delta \text{ Var } \dot{X}_{N-1}^S$$

(b) Calculate the non-maneuver correlation gate

$$C_N = 3.0 \; (\text{Var } X_N^P + \text{Var } X_N^m)^{\frac{1}{2}}$$

(c) Calculate the weighting coefficients

$$\alpha_N = \frac{\text{Var } X_N^P}{\text{Var } X_N^P + \text{Var } X_N^m} \; ; \quad \beta_N = \frac{\text{Cov } X\dot{X}_N^P}{\text{Var } X_N^P + \text{Var } X_N^m}$$

(d) Calculate an improved estimate of the track parameters

$$X_N^S = X_{N-1}^S + \dot{X}_{N-1}^S \Delta + \alpha_N \; (X_N^m - X_N^P)$$

$$\dot{X}_N^S = \dot{X}_{N-1}^S + \beta \; (X_N^m - X_N^P)$$

(e) Calculate the smoothed covariance elements

$$\text{Var } X_N^S = \alpha_N \text{ Var } X_N^m$$

$$\text{Var } \dot{X}_N^S = \text{Var } \dot{X}_N^P - \beta_N \text{ Cov } X\dot{X}_N^P$$

$$\text{Cov } X\dot{X}_N^S = \beta_N \text{ Var } X_N^m$$

Table IV shows values for the positional weighting coefficient α, the speed weighting coefficient β and the correlation distance C for 30 iterations of the recursive tracking equations under the assumption of unity blipscan and constant variance measurement data. Variance is constant since the aircraft is assumed to continue flying along the X axis, so that Var X^m equals Var R. It is perhaps of interest that under these assumptions, successive weighting parameters α and β may be obtained from the relationships

$$\alpha = \frac{2 \; (2 \; N + 1)}{(N + 1) \; (N + 2)}$$

for $N \geq 2$ = iteration number $\qquad$ (33)

$$\beta = \frac{0.6}{(N + 1) \; (N + 2)}$$

Table IV

Unity Blipscan Constant Variance Parameters

N	Positional Weighting Coefficient α	Speed Weighting Coefficient β	Correlation Distance C
1	1.0	.1000	–
2	.83	.0500	1.84
3	.70	.0300	1.37
4	.60	.0200	1.19
5	.52	.0143	1.09
6	.46	.0107	1.02
7	.42	.0083	.98
8	.38	.0067	.95
9	.34	.0054	.93
10	.32	.0045	.91
11	.29	.0038	.89
12	.27	.0033	.88
13	.26	.0028	.87
14	.24	.0025	.86
15	.23	.0022	.85
20	.18	.0013	.83
30	.12	.0006	.80

N is the iteration number

Examination of equations (33) and Table IV reveals that the positional and speed weighting coefficients converge to zero under the assumptions of unity blipscan and constant variance. This occurs because the model of aircraft motion that was used to derive the Kalman filter equations assumed constant speed, straight line motion. In turn, the derived algorithm assumes that its estimates of the track's parameters are improving as each measurement datum is processed until eventually its estimates achieve action, thereby requiring no further measurement data.

In the real world and for the assumed conditions, α and β would not converge to zero because the correlation area would become so small that even measurement data from aircraft which are attempting to follow the assumed model of motion would fall outside of the non-maneuver correlation gates. Upon detecting this condition, as will later be shown, the Kalman filter would be reinitiated to approximately its initialized state and the counting down process would again commence. To prevent the attendant oscillations associated with this technique, which is more a lack of technique, and to maintain a reasonable level of response to measurement data so that the tracker may rapidly respond to maneuvers, the usual approach is to set lower bounds on the dynamically computed weighting coefficients and correlation distances.

Rather than dynamically calculating weighting coefficients and the correlation distances, classical tracking algorithms have used prestored values for these parameters. Typical non-maneuver values for the positional weighting coefficient α, the speed weighting coefficient β and the correlation distance C are

$$\alpha = 0.3$$

$$\beta = 0.004$$

$$C = 1.5 \text{ NM}$$

These values of α and β represent suitable steady state values or lower bounds for a Kalman filtering algorithm. From Table IV, it is seen that these parameters are attained by the tenth iteration. Assuming a ten second iteration interval and a unity blipscan ratio, steady state parameters for a track are attained after approximately a minute and a half of tracking.

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Unity Blipscan Y Axis Aircraft Motion

The previous example was based upon unity blipscan constant variance measurement data. Let us now consider another example where two aircraft are flying along the Y axis of a rectilinear coordinate system centered at the radar site. Both aircraft are assumed to be initiated 200 NMs from the radar site, with one aircraft proceeding at a speed of 1.0 NM per 10 second iteration or 360 knots and the other at a rate of 4 NM per 10 second iteration or 1440 knots.

From equation (18) and assuming that the measurement data associated with each aircraft are reported at an azimuth of $\psi = 0$ degrees, it will be noted that the variance of the X component of any measurement datum is proportional to the product R^2 Var θ. Therefore, unlike the previous example where the standard deviation of each measurement datum was a constant 0.25 NM, in this example the standard deviation of measurement data associated with the aircraft is not constant and varies as a function of range from the radar site. The standard deviation of azimuth jitter will be assumed to equal 3 ACPs.

Rather than presenting the individual computations as was accomplished for the previous example, Table V shows values for the positional weighting coefficient α, the speed weighting coefficient β and the correlation distance C for 30 iterations of the recursive Kalman equations for both the 360 and 1440 knot aircraft under the assumption of unity blipscan and non-constant variance measurement data. The variance of each measurement datum is based upon the range of the aircraft with which it associates; i.e., at the tenth iteration, the range of the measurement datum associated with the aircraft proceeding at 1440 knots or 4 NM per iteration is 160 NM. By way of comparing the resultant data with the constant variance example of Table IV, values of α, β and C from Table IV will again be presented in Table V.

Examination of the three columns of data under α indicates that the positional weighting coefficient is essentially independent of the aircraft's location and of whether the variance of the measurement data is constant or not. Examining the three columns of data under the speed weighting coefficient β indicates that a similar statement may be made for this parameter. Table V also shows that for a unity blipscan ratio and for any of the three examples given, the weighting coefficients attain steady state values corresponding to the non-maneuver weighting coefficients of classical tracking algorithms in approximately ten iterations.

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Table V

Unity Blipscan, Non-Constant Variance Parameters

N	Positional Weighting Coefficient α			Speed Weighting Coefficient β			Correlation Distance C		
	CV	V1	V2	CV	V1	V2	CV	V1	V2
1	1.0	1.0	1.0	.1000	.1000	.1000	–	–	–
2	.83	.83	.84	.0500	.0501	.0503	1.84	6.76	6.74
3	.70	.70	.71	.0300	.0301	.0306	1.37	5.01	4.93
4	.60	.60	.61	.0200	.0202	.0207	1.19	4.32	4.18
5	.52	.53	.54	.0143	.0144	.0150	1.09	3.94	3.76
6	.46	.47	.48	.0107	.0109	.0114	1.02	3.70	3.47
7	.42	.42	.44	.0083	.0085	.0090	.98	3.52	3.25
8	.38	.38	.40	.0067	.0068	.0073	.95	3.39	3.08
9	.34	.35	.37	.0054	.0056	.0061	.93	3.29	2.93
10	.32	.32	.35	.0045	.0047	.0052	.91	3.21	2.81
11	.29	.30	.33	.0038	.0040	.0044	.89	3.14	2.69
12	.27	.28	.31	.0033	.0034	.0039	.88	3.08	2.59
13	.26	.26	.29	.0028	.0030	.0034	.87	3.03	2.50
14	.24	.25	.28	.0025	.0026	.0030	.86	2.98	2.41
15	.23	.24	.27	.0022	.0023	.0027	.85	2.94	2.32
20	.18	.19	.22	.0013	.0014	.0018	.83	2.77	1.94
30	.12	.13	.19	.0006	.0007	.0011	.80	2.53	1.29

N is the iteration number.

CV is the constant variance example of X axis aircraft motion.

V1 is the non-constant variance 360 knot example of Y axis aircraft motion.

V2 is the non-constant variance 1440 knot example of Y axis aircraft motion.

Examination of the first column of data under the correlation distance C shows non-maneuver gates that are smaller than those of classical tracking algorithms. This occurs because the aircraft is assumed to be proceeding along the X axis, where the variance of the associated measurement data is only affected by the data's range jitter, the standard deviation of which is assumed to be .25 NM.

Examination of the last two columns of data under the correlation distance C reveals non-maneuver gates that are generally larger than those of classical tracking algorithms with the last column of data decreasing more rapidly than the preceding column. The values are larger because the two aircraft are assumed to be proceeding along the Y axis, where the standard deviation of the associated measurement data is affected by the product of range times the data's azimuth jitter, the standard deviation of which is assumed to be 3 ACPs. Furthermore, the two aircraft are at a considerable range from the radar site. If the two aircraft were initiated 100 NM from the site rather than 200 NM, each of the correlation distances in the last two columns of data would be reduced by a factor of two; similarly, a 50 NM initiation process would decrease the data by a factor of four. The correlation distances in the last column of data under C decrease more rapidly than the correlation distances in the preceding column of data because the aircraft is approaching the radar site more rapidly thereby generating smaller range measurements.

Non-Unity Blipscan Ratio

Each of the previous examples has considered unity blipscan ratio situations where a measurement datum is available each iteration of the tracking algorithm. Although this situation may be approximated in netted radar systems where multiple radars provide measurement data on aircraft, lesser blipscan ratios are the norm in single radar systems.

Examination of equations (20) and (22) indicates that the correlation distance C and the position and speed weighting coefficients α and β are based upon the variance of the predicted track position Var X^P. From equation (19), we see that Var X^P increases as the time interval Δ between smoothings increases. Therefore, we should expect that α, β and C will increase whenever data are missed in a radar scan. This is reasonable because the reliability of the positional estimate of an aircraft decreases as the time between smoothing increases. In turn, the weighting coefficients should increase so the measurement datum will more strongly affect the new estimate of the aircraft's parameters. Similarly, the correlation distance should increase to ensure that measurement data associated with the aircraft correlates in the following radar scan. To verify this intuitive judgment, several examples of the effect of non-unity blipscan ratios on the weighting coefficients and correlation distances will be presented.

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Table VI shows the effect of missing data in the fifth scan on the weighting coefficients α and β and the correlation distance C. For this and the following examples within this section, the X variance of the measurement data is assumed to be 1.0 NM. When data is missing in a radar scan, no iteration of the recursive formulas occurs.

Table VI

Effect of Non-Unity Blipscan Ratio - Case I

	Scan	N	α	β	C
Unity Blipscan Ratio	4	4	.60	.020	4.7
	5	5	.52	.014	4.3
	6	6	.46	.011	4.1
	7	7	.42	.008	3.9
Miss in Scan 5	4	4	.60	.020	4.7
	6	5	.64	.028	5.0
	7	6	.49	.009	4.2

N is the iter ion number

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Table VII shows the effect of missing data in the thirteenth scan on α, β and C.

Table VII

Effect of Non-Unity Blipscan Ratio – Case II

	Scan	N	α	β	C
	12	12	.27	.003	3.5
	13	13	.26	.003	3.5
Unity Blipscan Ratio					
	14	14	.24	.003	3.4
	15	15	.23	.002	3.4
	12	12	.27	.003	3.5
Miss in Scan 13	14	13	.30	.006	3.6
	15	14	.27	.003	3.5

Table VIII shows the effect of missing data in four scans on the computation of α, β and C.

Table VIII

Effect of Non-Unity Blipscan Ratio – Case III

	Scan	N	α	β	C
	5	5	.52	.014	4.3
	6	6	.46	.011	4.1
	7	7	.42	.008	3.9
Unity Blipscan Ratio	8	8	.38	.007	3.8
	9	9	.34	.005	3.7
	10	10	.32	.004	3.6
	11	11	.29	.004	3.6
	5	5	.52	.014	4.3
Miss in Scans 6 to 9	10	6	.77	.049	6.3
	11	7	.49	.006	4.2

Examination of the data in Tables VI through VIII reveals that the correlation distances and weighting factors as calculated by a Kalman filter are appreciably affected by the blipscan ratio of the system's sensors.

MANEUVER RESPONSE

For any class of aircraft maneuver, a Kalman filter that will
track the aircraft while it executes the postulated maneuver may be
designed. However, such designs usually result in complex algorithms
involving many state variables with extraordinary computer processing
and storage requirements. Consequently, aircraft tracking algorithms
are typically designed based upon a constant speed straight line
model of aircraft motion as has previously been stated. To provide
for situations where the aircraft motion deviates from this model,
maneuver response logic are included in the algorithm.

The usual procedure upon detecting a maneuver in a Kalman tracking
filter is to approximate the initialization of the filter; upon cor-
relating a return and identifying an aircraft maneuver, the smoothed
track position is relocated to approximately the coordinates of the
radar report that stimulated the maneuver response. Similarly, the
elements of the smoothed covariance matrix P^S are increased to values
that approximate their initialized values as presented in equation (17).
The effect of this reinitialization is to place greater credibility in
the measurement data so that the filter will adequately track the
maneuvering aircraft.

A more sophisticated maneuver response logic would examine the
relationship between consecutive predicted track locations and the
associated measurement data to determine whether the aircraft is
turning at a constant speed or changing speed. Based upon this deter-
mination, some or all of the elements of the smoothed covariance matrix
would be set to larger values.

Independent of whether the approach is rudimental or sophisticated,
maneuver response in a Kalman filter is accomplished by resetting the
weighting coefficients α and β to values that approximate unity and by
resetting some or all components of the smoothed covariance matrix to
larger values. The intended effect is to place less credibility in
the predicted track parameters and more credibility in the data that
triggered the maneuver response so that the track will adequately
follow the aircraft maneuvers.

Possible users are cautioned that the doctrinaire implementation
of the above described maneuver logic may well lead to severe velocity
transients in a real world tracking situation. In a noise free environ-
ment and with error free measurement data, the above described maneuver
logic will result in a track that responds to maneuvers quite rapidly.
In a real world tracking environment and with a radar that provides a
noisy R, ψ measurement of an aircraft less than once per radar scan,

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this logic will not follow maneuvers in the steady, smooth manner of classical algorithms. Instead, violent changes in the track's parameters will occur as the track responds to the maneuver returns. Similarly, as the positional and velocity maneuver weighting coefficients approximate unity, excessive track instabilities will result when noise returns are erroneously correlated with a track.

SECTION IV

KALMAN FILTERING CONSIDERATIONS

SYSTEM MODELING

The application of Kalman filtering theory requires the definition
of a linear mathematical model of the system. In many cases, highly
complex models may be derived to accurately describe a system even
though only a few of the state variables are of primary interest.
Since computer storage and processing requirements increase substan-
tively with the complexity of the system model, designers should be
motivated to seek simplifications to the modeling equations that will
result in minimal degradation in the estimation of the major state
variables. Choice of coordinate systems and state variables should be
heavily influenced by the simplicity of the resultant formulation. In
summary, judicious modeling that is sufficiently complex to satisfy
system requirements while minimizing the associated processing and
storage requirements is the goal.

A related consideration in applying Kalman filter theory is the
impact of the lack of a precise knowledge of an exact model of the
system's behavior. For example, in determining the orbit of a satel-
lite, various unknown forces such as solar pressure, fuel leakage, etc.
may be affecting the true path of the satellite. If only the initial
injection positional and velocity uncertainties are modeled, the
estimated orbit will diverge from the true path of the satellite.
This results because the filter that is derived from an imprecise
model assumes that the estimated orbital parameters are converging
to their true values as successive measurement data are processed.
Based upon the erroneous assumption of convergence, weighting coeffi-
cients are chosen so that successive measurement data have lessened
impact on the estimated orbital parameters, the correlation area is
diminished and eventually measurement data fails to correlate. Some
techniques that are typically employed to compensate for inadequacies
in the system model are:

(a) To introduce pseudo-errors which cause an increase
in the values of the a priori covariance matrix to
account for non-modeled errors.

(b) Recognizing that the values of the a priori covari-
ance matrix are optimistic or too small, overweight
the more recent data relative to an optimal filter.

(c) To prevent the elements of the covariance matrix
and in turn the derived weighting coefficients
and correlation distance from decreasing below
some pre-established lower bound.

The use of a lower bound is particularly appropriate to an application
of Kalman filtering such as aircraft tracking when the filter is
derived from a model of constant speed straight line motion. For
unlike a spacecraft or terrestrial satellite, which can generally be
accurately predicted for weeks, the future position of an aircraft
can only be predicted accurately for a few seconds. Consequently, an
aircraft tracking filter must consistently remain relatively respon-
sive to the measurement data which implies that the elements of the
covariance matrix and the weighting coefficients should be prevented
from approaching zero.

A final modeling consideration in applying Kalman filter theory
is that a precise knowledge of the a priori statistics associated with
the predicted state variables may not be known. In such cases, con-
servative error estimates that are larger than the true errors are
typically used. The effect of employing excessive error estimates
with the initial predicted state variables is to increase the errors
associated with successive estimates of the state variables which in
turn lengthens the convergence interval. On the other hand, the use
of optimistic or small error estimates associated with the initial
predicted state variables prevents the filter from using adequate
weighting coefficients, which are sometimes referred to as the Kalman
gain, during the important initial period of estimation.

PROCESSING AND STORAGE REQUIREMENTS

Kalman filtering algorithms derived from complex system models
can impose extraordinary storage and processing requirements. There-
fore, various schemes which provide the essential benefits of Kalman's
approach but which minimize demands upon the processor should be
explored. In addition to simplifying the modeling equations, which
has already been discussed, other compromises are possible.

One such technique is to partition the state variables into
several groups. The Kalman theory is then independently applied to
the state variables of the lower dimensional groups and the estimate
of the entire set of state variables is reconstructed from the lower
dimensional groups. A caveat relative to such partitioning is to
assign state variables that are highly correlated to the same group
since partitioning essentially assumes a lack of correlation between
elements of different groups. For example, in the tracking example
that was discussed in both Section III and Appendix B, a sufficient
set of state variables which entirely described the system was X, Ẋ,

Y and $\dot{Y}$. These state variables were partitioned into a group X, $\dot{X}$ and into another group Y, $\dot{Y}$. An X component filter algorithm and a Y component filter algorithm were then derived according to the Kalman theory. Having derived the algorithms, a measurement datum was used with the X component filter equations to obtain an improved estimate of X and $\dot{X}$. Thereafter, the same measurement datum was used with the Y component filter equations to obtain an improved estimate of Y and $\dot{Y}$. Then, the entire set of state variables X, $\dot{X}$, Y and $\dot{Y}$ were reconstructed from the lower dimensional groups X, $\dot{X}$ and Y, $\dot{Y}$. Partitioning in this manner results in two similar Kalman problems that must be solved each iteration but it may be shown that this approach is more conservative in processing and storage requirements because the cross correlations between groups are ignored.

Another technique that should be explored to decrease processing demands is the precalculation and storage of the weighting coefficients and correlation distances or areas. This technique has practical importance in that it is possible to trade a small amount of storage space for considerable processing time. The technique is particularly well suited to applications where:

(a) The random noise that is superimposed upon the measurement data is relatively constant, and

(b) The lower bound, that is imposed upon the elements of the covariance matrix and in turn the weighting coefficients and correlation distances, is attained in a few iterations.

KALMAN VERSUS CONVENTIONAL FILTERING

The essential difference between Kalman's and more conventional filtering techniques is the real time propagation of statistical error measures of the state variables. These error measures are used in conjunction with the measurement datum's error measures to determine whether the datum should be correlated; if correlated, the error measures are used to calculate the gain to be applied to the measurement datum in determining an improved estimate of the parameters of interest. Conventional approaches generally involve the use of pre-stored correlation limits and gains.

Kalman filtering theory is described in precise mathematical terms and as such is perhaps more intellectually rewarding than applying more conventional techniques. However, possible users are cautioned that considerable experience is required in developing an effective filter because of the difficulties in defining the major state variables, obtaining knowledge of their statistical parameters, adequately modeling the system and striking a reasonable compromise

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between performance, storage and processing requirements. Relative to real-time aircraft tracking applications, the model of motion becomes especially difficult to formulate and typical Kalman trackers therefore assume constant speed straight line aircraft motion. Since such a filter is incapable of tracking a maneuvering aircraft, Kalman trackers are augmented with maneuver response logics that approximate conventional maneuver logics.

Upon developing a Kalman tracker, users are further cautioned that its implementation in an operational system is not a precise science. Doctrinaire implementations will generally result in practical difficulties, the resolution of which will tend to revert the process to more conventional techniques. Consequently, careful reflection on Kalman filters and their resultant performance and effectiveness relative to more classical filters is suggested prior to their implementation. Although well suited to extra-terrestrial tracking problems where equations of motion are readily formulated, the Kalman filter is not nor was it ever intended to be a panacea for any filtering application.

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APPENDIX A

ONE DIMENSIONAL KALMAN FILTER

INTRODUCTION

This Appendix presents a derivation of the one dimensional Kalman filter equations based only upon the concepts of the mean value and standard deviation of a random variable. This derivation comprises the majority of the elements of the complete Kalman filter except that it is a one dimensional, which does not require matrix manipulation, and there are no dynamics associated with the problem. Having derived the one dimensional filter equations for the initial iteration, the recursive feature of the filter equations will then be shown.

OPTIMAL COMBINATION OF INDEPENDENT ESTIMATES

The Kalman filtering process consists of combining two independent estimates of a variable in order to obtain an improved estimate. The improved estimate or weighted mean is obtained by updating the previous estimate based upon measurement data and the variances of both the previous estimate and the measured datum. The weighting factor that is chosen to yield an improved estimate with minimum variance and hence maximum probability is derived below.

Given the predicted range value of a point target R^P with variance σ_P^2 and an associated range measurement R^M with variance σ_R^2, these two estimates may be optimally combined to yield a minimum variance smoothed estimate of the point target's range.

The general form of a smoothed estimate R^S of R^P and R^M, where the value of α must be determined, is

$$R^S = R^P + \alpha(R^M - R^P) = (1 - \alpha)\ R^P + \alpha R^M \tag{1}$$

The expected or mean value of R^S, written $E(R^S)$, is

$$E(R^S) = (1 - \alpha)\ E(R^P) + \alpha E(R^M) \tag{2}$$

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The variance of R^S, written Var R^S is defined by

$$\text{Var } R^S = E\{R^S - E(R^S)\}^2$$

$$= E\{(1-\alpha)R^P + \alpha R^M - (1-\alpha)E(R^P) - \alpha E(R^M)\}^2$$

$$= E\{(1-\alpha)(R^P - E(R^P)) + \alpha(R^M - E(R^M))\}^2 \qquad (3)$$

$$\text{Var } R^S = (1-\alpha)^2 \text{ Var } R^P + \alpha^2 \text{ Var } R^M$$

since $E\{(R^P - E(R^P))(R^M - E(R^M))\} = 0$ as R^P and R^M are independent estimates.

To determine the value of α for which Var R^S is a minimum, the expression for Var R^S will be partially differentiated with respect to α and set to zero.

$$\frac{\partial \text{ Var } R^S}{\partial \alpha} = -2(1-\alpha) \text{ Var } R^P + 2\alpha \text{ Var } R^M = 0$$

$$\alpha(\text{Var } R^P + \text{Var } R^M) = \text{Var } R^P$$

yielding the optimum value of α for a weighting coefficient as

$$\alpha = \frac{\text{Var } R^P}{\text{Var } R^P + \text{Var } R^M} \qquad (4)$$

Substituting this value of α in equation (3) yields

$$\text{Var } R^S = \frac{\text{Var } R^P \text{ Var } R^M}{\text{Var } R^P + \text{Var } R^M} = (1-\alpha) \text{ Var } R^P \qquad (5)$$

Equations (1), (4) and (5) constitute the one dimensional Kalman filter equations. To initialize these equations, it is necessary to assign a priori numerical values to the predicted range R^P, to the variance of the predicted range Var R^P and to the variance of each measurement datum Var R^M. With values for these parameters and an initial measurement datum, equation (4) may be solved to determine the appropriate weighting coefficient, equation (1) may be solved to determine the smoothed estimate of the point target's range and equation (5) may be solved to determine the variance of the smoothed range estimate. This process is described in Section II of the report.

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In the absence of any a priori knowledge of the predicted range R^P and the variance of the predicted range Var R^P, the Kalman filter equations may be initialized by setting

$$\alpha_1 = 1.0$$

$$R_1^S = R_1^M$$

$$Var\ R_1^S = Var\ R_1^M$$

RECURSIVE FEATURE OF THE KALMAN FILTER

When another range measurement is obtained, the recursive feature of the Kalman filter allows the initial smoothed estimate R^S to be improved based upon the measurement data R^M and a different weighting coefficient. Like the previous smoothed estimate, this improved estimate will have minimum variance and hence maximum probability of representing the true range value. From the previous paragraph and with the subscript 1 added to indicate the first iteration, the initial set of recursive equations is

$$\alpha_1 = \frac{Var\ R_1^P}{Var\ R_1^P + Var\ R_1^M} \tag{6}$$

$$R_1^S = R_1^P + \alpha_1(R_1^M - R_1^P) \tag{7}$$

$$Var\ R_1^S = (1 - \alpha_1)\ Var\ R_1^P \tag{8}$$

Since the target is assumed to be stationary, the smoothed variance from the first computation becomes the predicted variance for the next computation. Similarly the smoothed range estimate from the first computation becomes the predicted range estimate for the following computation. Therefore, the recursive equations for the following computation become

$$\alpha_2 = \frac{Var\ R_1^S}{Var\ R_1^S + Var\ R_2^M} \tag{9}$$

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$$R_2{}^S = R_1{}^S + \alpha_2(R_2{}^M - R_1{}^S) \tag{10}$$

$$\text{Var } R_2{}^S = (1 - \alpha_2) \text{ Var } R_1{}^S \tag{11}$$

Each time another range measurement is obtained, a new weighting coefficient α is computed. It is applied against both the most recent range estimate and the measured datum to obtain an improved range estimate. Thereafter, the variance associated with the improved estimate is computed. This variance is required to calculate the weighting coefficient that will be used in the following iteration.

The general recursive formulas are:

(a) Calculate the weighting coefficient

$$\alpha_N = \frac{\text{Var } R_{N-1}{}^S}{\text{Var } R_{N-1}{}^S + \text{Var } R_N{}^M}$$

(b) Use the weighting coefficient to obtain an improved estimate

$$R_N{}^S = R_{N-1}{}^S + \alpha_N(R_N{}^M - R_{N-1}{}^S)$$

(c) Calculate the variance of the improved estimate

$$\text{Var } R_N{}^S = (1 - \alpha_N) \text{ Var } R_{N-1}{}^S$$

To commence or initialize this cyclic process three a priori numerical quantities are required. They are the predicted value of the range prior to taking any measurements, the variance of the predicted range, and the variance of the range measurement. In the absence of a priori knowledge of the predicted range and its associated variance, values for α_1, $R_1{}^S$ and Var $R_1{}^S$ may be assigned based upon the initial measurement datum. Thereafter, improved estimates are obtained as successive measurements are processed.

APPENDIX B

MULTI-DIMENSIONAL KALMAN FILTER

INTRODUCTION

Appendix A has derived one dimensional Kalman filter equations
for a problem that includes no dynamics. This Appendix augments that
material by extending the one dimensional filter theory to a multi-
dimensional system that includes dynamics.

A general form of Kalman's multi-dimensional, dynamic system
equations is stated. Each equation is then discussed in order to
provide a simple interpretation of its intent. Analogies between
the general Kalman equations and the one dimensional non-dynamic
filter that was derived in Appendix A are presented. No derivations
of the general Kalman filter equations are included since the deriva-
tions are either directly analogous to the development in Appendix A,
except that matrices are employed in place of scalars, or the deriva-
tions are readily accessible in most estimation theory textbooks that
have been written since 1960. Furthermore, lengthy matrix manipula-
tions are avoided in this manner and little comprehension of the
important aspects of the theory is sacrificed.

To adequately discuss the multi-dimensional, dynamic system
equations, the concepts of state variable or vector, transition
matrix, measurement matrix, covariance matrix and other notation of
state-space theory are introduced. However, rigorous definitions of
these terms, which most readers would in any event ignore, are avoided
as they too are readily available in numerous textbooks. In addition,
matrix derivations though elegant and efficient are suprisingly inef-
ficient in conveying an intuitive grasp of the salient features of the
theory. Therefore, the requisite state-space concepts associated
with the Kalman theory are presented via examples and in a manner
that is intended to reveal the important principles underlying Kalman's
estimation technique.

Having provided a discussion of the salient concepts associated
with each of the general Kalman equations, these equations are then
used to derive an algorithm for the real time tracking of piloted
aircraft.

GENERAL KALMAN FILTER EQUATIONS

A general form of the Kalman filter equations which are to be discussed and related to the one-dimensional filter equations are:

$$\text{Predicted State Variable} \qquad Z' = \emptyset Z^S \qquad\qquad (1)$$

$$\text{Predicted Covariance} \qquad P' = \emptyset P^S \emptyset^T \qquad\qquad (2)$$

$$\text{Measurement Equation} \qquad Z^m = MZ + N \qquad\qquad (3)$$

$$\text{Weighting Coefficient} \qquad S = P'M^T(MP'M^T + Q)^{-1} \qquad\qquad (4)$$

$$\text{Smoothed State Variable} \qquad Z^S = Z' + S\,(Z^m - MZ') \qquad\qquad (5)$$

$$\text{Smoothed Covariance} \qquad P^S = (I-SM)P' \qquad\qquad (6)$$

where:

Z, Z^S and Z' are the true, smoothed and predicted state variable column matrices

P^S and P' are the smoothed and predicted covariance matrices

M is the measurement matrix

S is the smoothing coefficient matrix

$\emptyset$ is the transition matrix

Z^m is the measurement column matrix

N is the measurement error column matrix

Q is the measurement error covariance matrix

I is the identity matrix

Predicted State Variable Equation

State variables may be defined as the minimum set of variables that provides full knowledge of a system's behavior. The position

and speed state variables for a constant speed straight line aircraft may be expressed in cartesian coordinates as

$$X^P = X^S + \dot{X}^S \Delta$$

$$\dot{X}^P = \ddot{X}^S$$

$$Y^P = Y^S + \dot{Y}^S \Delta$$ (7)

$$\dot{Y}^P = \dot{Y}^S$$

where the superscript P indicates the predicted state variables, where the superscript S indicates the smoothed state variables from the previous iteration and where Δ is the time since last smoothing. Equation (7) may be written in matrix notation as

$$\begin{bmatrix} X^P \\ \dot{X}^P \\ Y^P \\ \dot{Y}^P \end{bmatrix} = \begin{bmatrix} 1 & \Delta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X^S \\ \dot{X}^S \\ Y^S \\ \dot{Y}^S \end{bmatrix} \quad \text{or } Z' = \emptyset Z^S \quad (8)$$

Equation (8), the predicted state variable equation, is equivalent to equation (1) with the dynamics of the system being modeled by the matrix $\emptyset$ which is called the transition matrix.

The position, speed, and acceleration state variables for an aircraft may be expressed in cartesian coordinates as

$$X^P = X^S + \dot{X}^S \Delta + \ddot{X}^S \Delta^2/2$$

$$\dot{X}^P = \dot{X}^S + \ddot{X}^S \Delta$$

$$\ddot{X}^P = \ddot{X}^S$$

$$Y^P = Y^S + \dot{Y}^S \Delta + \ddot{Y}^S \Delta^2/2$$ (9)

$$\dot{Y}^P = \dot{Y}^S + \ddot{Y}^S \Delta$$

$$\ddot{Y}^P = \ddot{Y}^S$$

Equation (9) may be expressed in matrix notation as

$$
\begin{bmatrix} X^P \\ \dot{X}^P \\ \ddot{X}^P \\ Y^P \\ \dot{Y}^P \\ \ddot{Y}^P \end{bmatrix} = \begin{bmatrix} 1 & \Delta & \Delta^2/2 & 0 & 0 & 0 \\ 0 & 1 & \Delta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \Delta & \Delta^2/2 \\ 0 & 0 & 0 & 0 & 1 & \Delta \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X^S \\ \dot{X}^S \\ \ddot{X}^S \\ Y^S \\ \dot{Y}^S \\ \ddot{Y}^S \end{bmatrix} \quad \text{or } Z = \phi Z^S \qquad (10)
$$

Equation (8) represents a suitable one dimensional mathematical model for constant speed straight line aircraft motion and contains a 4 x 4 transition matrix because there are four state variables. Equation (10) represents an appropriate mathematical model for accelerating aircraft and contains a 6 x 6 transition matrix because of the addition of the acceleration state variable. As computer storage and particularly processing requirements increase rapidly with the number of state var. ~s or equivalently with the size of the transition matrix for Ka. ᵣ filters, designers are generally motivated to seek simplifications to the modeling equations that will result in minimum degradation in the estimation of the major state variables. For this reason, real time aircraft tracking filters usually employ a constant speed straight line model of aircraft motion as defined by equations (7) and (8) supplemented by a maneuver response logic.

In addition to simplifying the modeling equations by reducing the number of state variables other compromises, which further reduce the storage and processing requirements, are possible. One such technique is to partition the state variables into several groups as was discussed in Section III of this report.

A partitioned formulation of the X position and speed state variables for a constant speed straight line aircraft may be expressed in cartesian coordinates as

$$
X^P = X^S + \dot{X}^S \Delta
$$

$$
\dot{X}^P = \ddot{X}^S
$$

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This may be written in matrix notation as

$$\begin{bmatrix} X^P \\ \dot{X}^P \end{bmatrix} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X^S \\ \dot{X}^S \end{bmatrix} \quad \text{or } Z' = \phi Z^S$$

With the use of the transition matrix ϕ defined above and employing the general Kalman multi-dimensional equations, an X component tracking algorithm may be derived as will be shown. An independent but analogous Y component filter algorithm may similarly be derived. Use of independent, partitioned models of aircraft motion results in two analogous X and Y tracking algorithms that must be solved each iteration. However, it may be shown that this approach is more conservative in processing and storage requirements than simultaneously processing the four state variables because the cross correlations between groups such as XY, $\dot{X}\dot{Y}$ and $X\dot{Y}$ are ignored. The implications of these statements will become clearer when the X component Kalman tracking algorithm is formally derived.

Each of the aforementioned examples of equation (1), which is called the predicted state variable equation, has employed cartesian coordinates with X and Y as the state variables. Before proceeding to a discussion of another of the general Kalman equations, a final example of the predicted state variable equation will be presented that uses range and azimuth as the state variables. Selection of this coordinate system results in a predicted state variable equation that is similar to equation (8) and is described by

$$\begin{bmatrix} R^P \\ \dot{R}^P \\ \dot{\psi}^P \\ \dot{\psi}^P \end{bmatrix} = \begin{bmatrix} 1 & \Delta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R^S \\ \dot{R}^S \\ \dot{\psi}^S \\ \dot{\psi}^S \end{bmatrix} \quad \text{or } Z' = \phi Z^S \tag{11}$$

This polar formulation of the predicted state variable equation is generally incompatible with a constant speed straight line model of aircraft motion. Except for radial flight or constant speed circular flight about a radar site, an aircraft accelerates in a polar coordinate system even if it is flying in a straight line and at a constant speed. Therefore, polar formulations of the predicted state variable equation are generally avoided in applying Kalman filtering theory.

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Predicted Covariance Equation

In the one dimensional static filter equations that were derived in Appendix A, a fundamental input to the filter was the expected or predicted variance of range called Var R^p. A similar quantity called the covariance matrix is required for the multi-dimensional dynamic filter.

For the partitioned constant speed straight line cartesian coordinate formulation of aircraft motion, the smoothed covariance matrix P^S for the X component state variables has the following four elements

$$P^S = \begin{bmatrix} \text{Var } X^S & \text{Cov } X\dot{X}^S \\[2ex] \text{Cov } X\dot{X}^S & \text{Var } \dot{X}^S \end{bmatrix} \tag{12}$$

These elements are calculated each iteration after improved estimates of the state variables X and $\dot{X}$ have been obtained and are a measure of confidence in the current estimates of the state variables. The elements of the smoothed covariance matrix must be extrapolated to the time of the following iteration via equation (2), the predicted covariance equation, in order to calculate weighting coefficients for that iteration.

The variance of X^S and $\dot{X}^S$ written Var X^S and Var $\dot{X}^S$ are respectively defined as

$$\text{Var } X^S = E\{(X^S - E(X^S)) (X^S - E(X^S))\} = E (\epsilon X^S) (\epsilon X^S) = E(\epsilon X^S)^2$$

$$\text{Var } \dot{X}^S = E\{(\dot{X}^S - E(\dot{X}^S)) (\dot{X}^S - E(\dot{X}^S))\} = E (\epsilon \dot{X}^S) (\epsilon \dot{X}^S) = E(\epsilon \dot{X}^S)^2$$

The covariance of $X\dot{X}^S$ written Cov $X\dot{X}^S$ is defined as

$$\text{Cov } X\dot{X}^S = E\{(X^S - E(X^S)) (\dot{X}^S - E(\dot{X}^S))\} = E (\epsilon X^S) (\epsilon \dot{X}^S)$$

The terms ϵX^S and $\epsilon \dot{X}^S$ represent the errors in the variables X^S and $\dot{X}^S$; stated alternatively, ϵX^S represents the difference between the variable X^S and the mean of the variable X^S.

As with the one dimensional example,   ies for the elements of the covariance matrix must be known to ini' iii. the filter. These values may be determined by experimental data, from the measurement datum, by error analysis or by engineering jud . . The usual

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procedure in a tracking filter is to initialize the elements of the covariance matrix based upon the measurement data that determines the track's initial position and velocity.

The initial value of the variance of X^S is set equal to the variance of the X component of the measurement datum that is used in the initiation process. The variance of the X component of the measurement datum called Var X^m is based upon the measurement datum (R, ψ), a priori knowledge of both the variance in range Var R and the variance in azimuth Var θ, and is calculated from the following equation:

$$X^m = R \text{ Sin } \psi$$

$$X^m + \epsilon X^m = (R + \epsilon R)(\text{Sin } \psi + \epsilon \psi)$$

$$= (R + \epsilon R)(\text{Sin } \psi \text{ Cos } \epsilon \psi + \text{Cos } \psi \text{ Sin } \epsilon \psi) \qquad (13a)$$

$$= (R + \epsilon R)(\text{Sin } \psi + \epsilon \psi \text{ Cos } \psi) \text{ for small } \epsilon \psi$$

$$= R \text{ Sin } \psi + R \epsilon \psi \text{ Cos } \psi + \epsilon R \text{ Sin } \psi + \epsilon R \epsilon \psi \text{ Cos } \psi$$

Disregarding second order terms

$$\epsilon X^m = \epsilon R \text{ Sin } \psi + R \epsilon \psi \text{ Cos } \psi$$

$$(\epsilon X^m)^2 = (\epsilon R)^2 \text{ Sin}^2 \psi + R^2 (\epsilon \psi)^2 \text{ Cos}^2 \psi + 2(\epsilon R)(\epsilon \psi) R \text{ Sin } \psi \text{ Cos } \psi$$

As R and ψ are independent

$$\text{Var } X^m = E(\epsilon X^m)^2 = \text{Var R Sin}^2 \psi + R^2 \text{ Var } \theta \text{ Cos}^2 \psi$$

The initial value of the smoothed variance in X, given the X components of the two measurement data X_2^m and X_1^m used in the initiation process and the time difference Δ between the receipt of the two measurements, is obtained by noting that

$$\dot{X}^S = \frac{X_2^m - X_1^m}{\Delta}$$

$$\epsilon \dot{X}^S = \frac{\epsilon X_2^m - \epsilon X_1^m}{\Delta} \qquad (13b)$$

$$(\epsilon \dot{X}^S)^2 = \frac{(\epsilon X_2^m)^2 + (\epsilon X_1^m)^2 - 2(\epsilon X_2^m)(\epsilon X_1^m)}{\Delta^2}$$

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As X_1 and X_2 are independent

$$\text{Var } \dot{X}^S = E(\epsilon \dot{X}^S)^2 = \frac{2 \text{ Var } X^m}{\Delta^2}$$

The initial value of Cov $X\dot{X}^S$, given the X components of the two measurement data $X_2{}^m$ and $X_1{}^m$ used in the initiation process and the time interval Δ, is obtained from

$$X^S \dot{X}^S = X_2{}^m (X_2{}^m - X_1{}^m)/\Delta$$

$$\epsilon X^S \epsilon \dot{X}^S = \epsilon X_2{}^m (\epsilon X_2{}^m - \epsilon X_1{}^m)/\Delta \qquad (13c)$$

$$= \frac{(\epsilon X_2{}^m)^2 - (\epsilon X_2{}^m)(\epsilon X_1{}^m)}{\Delta}$$

As X_1 and X_2 are independent

$$\text{Cov } X\dot{X}^S = E(\epsilon X^S)(\epsilon \dot{X}^S) = \frac{\text{Var } X^m}{\Delta}$$

In the one dimensional example of Appendix A, the smoothed variance from one iteration was used as the predicted variance for the following iteration. This was perfectly reasonable since the error in range which ultimately determines the variance in range was only affected by the a priori initial conditions and the number of range measurement data that had been processed.

In the two dimensional tracking filter, the smoothed covariance elements from one iteration may not be used in the following iteration since the predicted error in the track's X position is affected by the previous error in the X position and the previous error in the X component of the track's speed multiplied by the time interval Δ. Equation 2, the predicted covariance equation, therefore calculates the predicted covariance matrix from the previously smoothed covariance matrix and the time interval from the previous to the current iteration. The predicted covariance matrix is obtained via the following relationship.

$$P' = \emptyset \, P^S \, \emptyset^T \qquad (14)$$

This equation indicates that the predicted covariance matrix P' is propagated from the smoothed covariance matrix P^S and $\emptyset$ squared. Equation (14), although not derived herein, appears reasonable; i.e., if $\emptyset$ propagates the state variables as stated in equation (1), then it is reasonable for $\emptyset$ squared to propagate the covariance of the state variables. The symbol $\emptyset^T$ in equation (14) means the transpose of $\emptyset$ and designates that the rows and columns of the matrix should be interchanged. Equation (14) is discussed via numerical examples in Section III of this report.

Measurement Equation

In the one dimensional example of Appendix A, each range measurement R^m was related to the true range R which was being estimated by the relationship

$$R^m = MR + \varepsilon R \tag{15}$$

where M was 1 and εR was the error in range. In a multi-dimensional problem, an analogous equation relate the measurement data to the true value of the state variables. Equation (3), the measurement equation is

$$Z^m = MZ + N \tag{16}$$

where Z^m and Z are the measurement column matrix and the true value(s) of the state variables respectively, and N is the measurement noise vector. Using the 2 x 2 partitioned cartesian coordinate formulation of the Kalman aircraft tracking filter, the X component equation corresponding to equation (16) is

$$X^m = X + \varepsilon X \tag{17}$$

In matrix notation, equation (17) may be expressed as

$$X^m = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ \dot{X} \end{bmatrix} + \varepsilon X \tag{18}$$

The matrix $M = \begin{bmatrix} 1 & 0 \end{bmatrix}$ is called the measurement matrix and simply indicates the component of the state variable that is being measured. The measurement matrix is $\begin{bmatrix} 1 & 0 \end{bmatrix}$ since the measured datum is a positional report; if the measured datum was a Doppler report, the measurement matrix would assume the form $\begin{bmatrix} 0 & 1 \end{bmatrix}$.

Considering the implications of the measurement equation further, if a 4 x 4 range and azimuth formulation of the Kalman filter is employed where each measurement datum is a positional report consisting of the parameters R^m and ψ^m, equation (16) would be equivalent to

$$R^m = R + \varepsilon R$$

$$\psi^m = \psi + \varepsilon,$$

(19)

In matrix notation, equation (19) may be expressed as

$$
\begin{bmatrix} R^m \\ \\ \psi^m \end{bmatrix} =
\begin{bmatrix} 1 & 0 & 0 & 0 \\ \\ 0 & 0 & 1 & 0 \end{bmatrix}
\begin{bmatrix} R \\ \dot{R} \\ \psi \\ \dot{\psi} \end{bmatrix} +
\begin{bmatrix} \varepsilon R \\ \\ \varepsilon \psi \end{bmatrix}
$$

As is evident, the measurement equation (16) relates any form of the measurement data to the state variables via the appropriate measurement matrix.

Weighting Coefficient Equation

In the one dimensional example of Appendix A, the measured range R^m was weighted with a smoothed estimate by a weighting coefficient α_N which was defined by

$$\alpha_N = \frac{\text{Var } R_{N-1}^S}{\text{Var } R_{N-1}^S + \text{Var } R_N^m}$$

(20)

An analogous express is employed in the multi-dimensional tracking problem, where the weighting coefficient S is defined from equation (4) by

$$S = P' M^T (M P' M^T + Q)^{-1}$$

(21)

Equation (21) will not be derived herein since its development exactly follows the derivation in Appendix A except that matrices are required in place of the scalars. The matrix manipulations are tedious but straightforward and may be found in most estimation theory textbooks.

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In equation (21), Q is the measurement error covariance matrix and is analogous to Var R^m in equation (20). For the tracking example that we are considering Q is equal to Var X^m. M is the measurement matrix as has previously been stated and is analogous to the factor unity in equation (20). P' is the predicted covariance matrix and is analogous to Var $R_{N-1}{}^S$. The superscript -1 indicates matrix inversion and is analogous to division. Ignoring the differences in notation between equations (20) and (21) and making use of the above stated analogies, it is apparent that equations (20) and (21) are identical. Consequently, understanding the implications of equation (20) for the simpler one dimensional filter example will yield a corresponding grasp of the implications of the multi-dimensional weighting coefficient S.

Smoothed State Variable and Covariance Equation

In the one dimensional example of Appenidx A, an improved estimate of range or $R_N{}^S$ was obtained from the expression

$$R_N{}^S = R_{N-1}{}^S + \alpha_N (R_N{}^m - R_{N-1}{}^S) \tag{22}$$

Furthermore, the variance of the improved estimate was calculated from the expression

$$\text{Var } R_N{}^S = (1 - \alpha_N) \text{ Var } R_{N-1}{}^S \tag{23}$$

Analogous expressions to equations (22) and (23) for the multi-dimensional problem from equations (5) and (6) are

$$Z^S = Z' + S(Z^m - MZ') \tag{24}$$

and

$$P^S = (I - SM) P' \tag{25}$$

The analogies between equations (22) and (24) as well as between (23) and (25) are sufficiently obvious so that no discussion is warranted. Again, an understanding of the implications of the one dimensional equations (22) and (23) will yield a corresponding level of comprehension for the matrix formulations of equations (24) and (25).

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TRACKING ALGORITHM

This section uses the general Kalman equations that have previously been presented and the transition matrix based upon a constant speed straight line model of aircraft motion to derive an algorithm for tracking aircraft. The derived Kalman filter algorithm is used for track smoothing, prediction and correlation. In addition to providing smoothed and predicted state variables which define the track's position and velocity, the Kalman filter algorithm provides estimates of the errors associated with the state variables.

Kalman Non-Maneuver Smoothing and Prediction

The track's state vector will be defined in terms of the variables X, $\dot{X}$, Y and $\dot{Y}$. The variables X and $\dot{X}$ will be partitioned from the variables Y and $\dot{Y}$. The covariance matrix for the X variables consists of the terms Var X, Var $\dot{X}$ and Cov $X\dot{X}$. Since the X and Y filters are partitioned and hence completely uncoupled, it will only be necessary to derive the tracking algorithm for the X component. Replacing X by Y will yield an analogous Y component filter.

As previously indicated the state variable Z, the measurement matrix M, the measurement column matrix Z^M and the measurement covariance matrix Q are defined by

$$Z = \begin{bmatrix} X \\ \dot{X} \end{bmatrix} \quad ; \quad M = [1 \quad 0] \quad ; \quad Z^m = X^m \text{ and } Q = \text{Var } X^m$$

From equation (8), the transition matrix corresponding to a constant speed straight line model of aircraft motion is defined by

$$\emptyset = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix}$$

where Δ is defined as the time interval since last track smoothing. The elements of the predicted covariance matrix P' and the smoothed covariance matrix P^S are defined by

$$P' = \begin{bmatrix} \text{Var } X^P & \text{Cov } X\dot{X}^P \\ \text{Cov } X\dot{X}^P & \text{Var } \dot{X}^P \end{bmatrix} \quad ; \quad P^S = \begin{bmatrix} \text{Var } X^S & \text{Cov } X\dot{X}^S \\ \text{Cov } X\dot{X}^S & \text{Var } \dot{X}^S \end{bmatrix}$$

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The elements of the weighting coefficient S which will become the positional and velocity smoothing coefficients are defined by

$$S = \begin{bmatrix} \alpha \\ \\ \beta \end{bmatrix}$$

From equation (2) and the definitions of $\emptyset$, P^S and $\emptyset^T$, the elements of the predicted covariance matrix P' are determined as follows

$$\begin{bmatrix} \text{Var } X^P & \text{Cov } X\dot{X}^P \\ \text{Cov } X\dot{X}^P & \text{Var } \dot{X}^P \end{bmatrix} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \text{Var } X^S & \text{Cov } X\dot{X}^S \\ \text{Cov } X\dot{X}^S & \text{Var } \dot{X}^S \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \Delta & 1 \end{bmatrix}$$

Performing the requisite matrix multiplications, the elements of the predicted covariance matrix P' are determined to be

$$\text{Var } X^P = \text{Var } X^S + 2\Delta \text{ Cov } X\dot{X}^S + \Delta^2 \text{ Var } \dot{X}^S$$

$$\text{Var } \dot{X}^P = \text{Var } \dot{X}^S \qquad\qquad (26)$$

$$\text{Cov } X\dot{X}^P = \text{Cov } X\dot{X}^S + \Delta \text{ Var } \dot{X}^S$$

Equations (26) are used to extrapolate the smoothed covariance matrix from one iteration to the time of the next track smoothing in the following iteration.

From equation (4) and the appropriate definitions, the elements of the weighting coefficient S are determined as follows

$$S = P'M^T(MP'M^T + Q)^{-1}$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \text{Var } X^P & \text{Cov } X\dot{X}^P \\ \text{Cov } X\dot{X}^P & \text{Var } \dot{X}^P \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left\{ (1 \quad 0) \begin{bmatrix} \text{Var } X^P & \text{Cov } X\dot{X}^P \\ \text{Cov } X\dot{X}^P & \text{Var } \dot{X}^P \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \text{Var } X^m \right\}^{-1}$$

Solving for α and β, the elements of the weighting coefficient S are determined to be

$$\alpha = \frac{\text{Var } X^P}{\text{Var } X^P + \text{Var } X^M} \quad ; \quad \beta = \frac{\text{Cov } X\dot{X}^P}{\text{Var } X^P + \text{Var } X^M} \qquad (27)$$

From equation (5) and the appropriate definitions, the smoothed state variable Z^S, the components of which are X^S and $\dot{X}^S$, is determined as follows

$$Z^S = Z' + S(Z^m - MZ')$$

$$\begin{bmatrix} X^S \\ \dot{X}^S \end{bmatrix} = \begin{bmatrix} X^P \\ \dot{X}^P \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \left\{ X^m - [1 \quad 0] \begin{bmatrix} X^P \\ \dot{X}^P \end{bmatrix} \right\}$$

Performing the required matrix arithmetic, the elements of the smoothed state variables are determined to be

$$X^S = X^P + \alpha \, (X^M - X^P)$$

$$\dot{X}^S = \dot{X}^P + \beta \, (X^M - X^P)$$

(28)

Finally, from equation (6) and the appropriate definitions, the smoothed covariance matrix P^S, the components of which are Var X^S, Cov $X\dot{X}^S$ and Var $\dot{X}^S$, is determined as follows

$$P^S = (I - SM) P'$$

$$\begin{bmatrix} \text{Var } X^S & \text{Cov } X\dot{X}^S \\ \text{Cov } X\dot{X}^S & \text{Var } \dot{X}^S \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \alpha \\ \beta \end{bmatrix} [1 \quad 0] \right\} \begin{bmatrix} \text{Var } X^P & \text{Cov } X\dot{X}^P \\ \text{Cov } X\dot{X}^P & \text{Var } \dot{X}^P \end{bmatrix}$$

Performing the matrix manipulations, the elements of the smoothed covariance matrix are determined to be

$$\text{Var } X^S = \alpha \text{ Var } X^m$$

$$\text{Cov } X\dot{X}^S = \beta \text{ Var } X^m$$

(29)

$$\text{Var } \dot{X}^S = \text{Var } \dot{X}^P - \beta \text{ Cov } X\dot{X}^P$$

Equations (26) through (29) constitute the X coordinate Kalman smoothing and prediction filter. Substituting Y for X in each of these equations results in the corresponding Y component filter equations.

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Tracker Initialization

For the one dimensional filter equations that were derived in Appendix A, three quantities were required in order to commence the iterative process. They were the expected or predicted value of the range prior to any measurement, the variance associated with the predicted range and the variance associated with the range measurement. As should be expected, analogous quantities are required in order to initialize the multi-dimensional tracking filter.

The quantities that are analogous to the expected value of range are the components of the state vector Z^S namely X^S and $\dot{X}^S$. Correspondingly to the variance of the predicted range are the elements of the smoothed covariance matrix P^S namely Var X^S, Cov $X\dot{X}^S$ and Var $\dot{X}^S$. Finally, analogous to the variance of the range measurement Var R^m are the cartesian coordinate variances associated with the (R, ψ) measurement datum.

Assuming a two point initiation process, the X components of the multi-dimensional quantities described above are defined as

$$X^S = X_2^m$$

$$\dot{X}^S = \frac{X_2^m - X_1^m}{\Delta} \tag{30}$$

where X_1^m, X_2^m are the X coordinates of the first and second measurement data associated with the two point initiation process. Δ is the time difference between the initial and the second R, ψ measurement datum.

Setting the initial value of Var X^S to the variance of the X component of the measurement datum used in the initiation process and from equations (13b) and (13c), the elements of the smoothed covariance matrix P^S are defined by

$$Var\ X^S = Var\ X^m$$

$$Var\ \dot{X}^S = \frac{2\ Var\ X^m}{\Delta^2} \tag{31}$$

$$Cov\ X\dot{X}^S = \frac{Var\ X^m}{\Delta}$$

Equation (31) states that the initial value of the variance of X^S is the variance of the X component of the measurement datum that is used in the initiation process. From equation (13a), Var X^m is determined from the relationship

$$\text{Var } X^m = \text{Var } R \, \text{Sin}^2 \, \psi + R^2 \, \text{Var } \theta \, \text{Cos}^2 \, \psi \tag{32}$$

where (R, ψ) are the coordinates of the measurement datum and where Var R and Var θ are based upon a priori knowledge of the sensor's characteristics.

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